Tutorial 3.   28 Sep 2015.

1. With $w = se^{i\varphi}$, where $s \geq 0$ and $\varphi \in \mathbb{R}$, Solve the equation $z^n = w$ in $\mathbb{C}$ where n is a natural number. How many solutions are there?

2. a). Let z, w be two complex numbers such that $\bar{z}w \neq 1$. Prove that

$$\left|\frac{w-z}{1-\bar{w}z}\right| < 1 \quad \text{if} \quad |z| < 1 \text{ and } |w| < 1.$$

and also that

$$\left|\frac{w-z}{1-\bar{w}z}\right| = 1 \quad \text{if} \quad |z| = 1 \text{ or } |w| = 1$$

[Hint: It is sufficient to Prove in the case that is real]

b) Prove that for a fixed w in the unit disc $\mathbb{D}$, the mapping

$$F : z \mapsto \frac{w-z}{1-\bar{w}z}$$

satisfies the following conditions:

(i) F maps the unit disc to itself (that is $F : \mathbb{D} \to \mathbb{D}$), and is holomorphic.

(ii) F interchanges 0 and w, namely $F(0) = w$ and $F(w) = 0$

(iii) $|F(z)| = 1$ if $|z| = 1$.

(iv) $F : \mathbb{D} \to \mathbb{D}$ is bijective. [Hint: Calculate $F \circ F$.]

3. Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases:

(a) $\mathrm{Re}(f)$ is constant;

(b) $\mathrm{Im}(f)$ is constant;

(c) $|f|$ is constant;

one can conclude that f is constant.

[Hint: Cauchy-Riemann Equations]